Field–dependent symmetries of a non-relativistic fluid model

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Abstract

As found by Bordemann and Hoppe and by Jevicki, a certain non–relativistic model of an irrotational and isentropic fluid, related to membranes and to partons, admits a Poincaré symmetry. Bazeia and Jackiw associate this dynamical symmetry to a novel type of "field–dependent" action on space–time. The "Kaluza-Klein type" framework of Duval et al. is used to explain the origin of these symmetries and to derive the associated conserved quantities. In the non-interacting case, the symmetry extends to the entire conformal group.

Revised version

Key words : membrane theory, fluid mechanics, symmetries and conservation laws.

1 Introduction

The reduction of membrane theory can lead to a simple model, describing an isentropic, irrotational fluid [1]. A similar system can be obtained, e. g., by dimensional reduction of relativistic field theory [2], and also in the hydrodynamical formulation of the (non–linear) Schrödinger equation [3]. The model, also used in gas dynamics, was further discussed by Bazeia, Jackiw, and Polychronakos [3], [4], [5], [6]; note also [7].

Let us consider the action

$$\mathcal{S} = \int dxdt \Big[\underbrace{- R\partial_t\Theta - \frac{1}{2}R(\partial_x\Theta)^2}_{\mathcal{L}_0} - V(R) \Big], \tag{1.1}$$

where $R(x,t) \geq 0$ and $\Theta(x,t)$ are real fields and $V(R)$ is some potential. (Our Lagrange density differs from that of Bazeia and Jackiw in [3] in a surface term; the two expressions are hence equivalent. Albeit similar results hold in any dimension, we shall restrict ourselves, for simplicity, to $(1+1)$ dimensionsal space–time, parametrized by position and time, x and t.).

The associated Euler-Lagrange equations read

$$\partial_t R + \partial_x(R\partial_x\Theta) = 0, \qquad \partial_t\Theta + \frac{1}{2}(\partial_x\Theta)^2 = -\frac{dV}{dR}. \tag{1.2}$$

In what follows, we shall (except in Section 7), restrict ourselves to potentials of the form $V = cR^\omega$, where c and ω are real constants. In the membrane case the effective potential is in particular

$$V(R) = \frac{c}{R}, \qquad c = \text{const}. \tag{1.3}$$

The Lagrangian (1.1) is first-order in the time derivative; it admits there-

fore an (extended) Galilean symmetry, with conserved quantities

$$
\begin{aligned}
H &= \int dx\, \underbrace{\left(\frac{1}{2}R(\partial_x\Theta)^2 + V(R)\right)}_{\mathcal{H}} && \text{energy}\\
P &= \int dx\, \underbrace{R\,\partial_x\Theta}_{\mathcal{P}} && \text{momentum}\\
B &= \int dx\,(xR - t\mathcal{P}) && \text{boosts}\\
N &= \int dx\, R && \text{particle number}
\end{aligned}
\tag{1.4}
$$

Unexpectedly, the free and the membrane systems both admit two additional conserved quantities [1], [2] [3], namely

$$
\begin{aligned}
G &= \int dx\,(x\mathcal{H} - \Theta\mathcal{P}) && \text{“antiboost”}\\
D &= tH - \int dx\, R\Theta && \text{time dilatation}
\end{aligned}
\tag{1.5}
$$

The generators (1.4)–(1.5) span furthermore the (2 + 1) dimensional Poincaré algebra [1], [2], [3]. The arisal of the typically *relativistic* Poincaré symmetry for a *non-relativistic system* is quite surprising. The mystery is increased by that this symmetry is *not* associated to any finite–dimensional group action on space-time. It belongs in fact to a new type of “field-dependent” non-linear action on space–time [3] which, to our knowledge, has never been met before.

Before explaining how these symmetries arise, we point out that, in the “free case” $V = 0$, the entire conformal group $\mathrm{O}(3,2)$ is a symmetry; it is reduced to the Poincaré group for $V(R) = c/R$, and to the Schrödinger group (the symmetry of the free Schrödinger equation [8]) for $V(R) = cR^3$, respectively.

Where do these symmetries come from ? We answer this question by unfolding the system into a higher–dimensional space, obtained by promoting the “phase” Θ to a “vertical” coordinate (we denote by s) on extended space, M. Such a “Kaluza–Klein–type” framework for non–relativistic physics was

put forward by Duval et al. [9]. In our case, their extended space M is $(2+1)$–dimensional Minkowski space, with x a spacelike and t and s light–cone coordinates. Then the strange, field–dependent, non–linear action of Bazeia and Jackiw [3], (Eq. (2.4) below), becomes the natural, linear action of the $(2+1)$–dimensional Poincaré group on extended space.

Our starting point is the simple but crucial observation due to Christian Duval [10] which says that, on extended space M, the "antiboosts" are the counterparts of galilean boosts, when galilean time, t, and the "vertical coordinate", s are interchanged,

$$t \longleftrightarrow s. \tag{1.6}$$

Many results presented in this paper come by exploiting this intechange–symmetry. For example,

- applied to the Galilei group, the Poincaré group is obtained;
- applied to "non-relativistic conformal symmetries" (Eq. (2.6) below) yields relativistic conformal symmetries,

etc. It also provides a clue for the non–conventional implementations on fields.

The action of the conformal group $\mathrm{O}(3,2)$ and its various subgroups on M is presented in Section 4. In Section 5 we project the natural, linear action on extended space to a "field-dependent action" on ordinary space. This requires to generalise as in Eq. (5.4) the usual equivariance condition (5.1) of Duval et al. [9]. The authors of Ref. [3] call the Poincaré symmetry "dynamical" since it is not associated to a natural "geometric" action on space–time. Our point is that these symmetries become "geometric" on extended space.

In Section 6 we study physics in the extended space and show how the previous results can be recovered. Our results show also that the "membrane potential"(1.3) i.e. $V(R) = c/R$ is the only one which can accomodate these new type of symmetries. This is the reason why these strange symmetries do *not* arise for the ordinary Schrödinger equation : this latter corresponds in fact to a particular effective potential, namely to $\overline{V} = -\frac{1}{8}\frac{(\vec{\nabla}R)^2}{R}$. Usual equivariance allows us in turn to recover the well-known Schrödinger symmetry.

In Ref. [6], the Poincaré symmetry of the fluid system (1.1) is related to that of the Nambu-Goto action of a membrane moving in higher dimensional space-time. Our "Kaluza–Klein" framework is an alternative way of obtaining the same conclusion.

2 Symmetries

We first recall the construction of the conserved quantities. Let us consider a non-relativistic theory given by the Lagrange density $\mathcal{L}(\partial_\alpha\phi, \phi)$, where ϕ denotes all fields collectively. Then Noether's theorem [11] says that if the Lagrange density changes by a surface term under an infinitesimal transformation $\phi \to \phi + \delta\phi$,

$$\delta\mathcal{L} = \partial_\alpha C^\alpha, \tag{2.1}$$

then $J^\alpha = \frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi)}\delta\phi - C^\alpha$ is a conserved current, $\partial_\alpha J^\alpha = 0$, so that

$$\int dx \left(\frac{\delta\mathcal{L}}{\delta(\partial_t\phi)}\delta\phi - C^t\right) \tag{2.2}$$

is conserved. For example, the usual Galilean transformations of non–relativistic space-time, $\begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} x^\star \\ t^\star \end{pmatrix}$, $\begin{pmatrix} R(x,t) \\ \Theta(x,t) \end{pmatrix} \to \begin{pmatrix} R^\star(x,t) \\ \Theta^\star(x,t) \end{pmatrix}$, where

$$\begin{array}{lll}
x^\star = x, & R^\star(x,t) = R(x, t+\tau), & \\
t^\star = t + \tau, & \Theta^\star(x,t) = \Theta(x, t+\tau); & \text{time translation} \\
 & & \\
x^\star = x - \gamma, & R^\star(x,t) = R(x-\gamma, t), & \\
t^\star = t, & \Theta^\star(x,t) = \Theta(x-\gamma, t); & \text{translation} \\
 & & \\
x^\star = x + \beta t, & R^\star(x,t) = \;\; R(x+\beta t, t), & \\
t^\star = t, & \Theta^\star(x,t) = \;\; \Theta(x+\beta t, t) & \text{boost} \\
 & \qquad\qquad\quad -\beta x - \frac{1}{2}\beta^2 t; & \\
 & & \\
x^\star = x & R^\star(x,t) = R(x,t), & \\
t^\star = t, & \Theta^\star(x,t) = \Theta(x,t) - \eta. & \text{phase shift}
\end{array} \tag{2.3}$$

change the Lagrange density (1.1) by a surface term, and Noether's theorem yields the conserved quantities (1.4). The new conserved quantities (1.5) belong in turn to the following strange, non–linear action on space-time [3]

$$\begin{array}{ll}
x^\star = x + \alpha\Theta(x^\star, t^\star) & \\
t^\star = t + \frac{1}{2}\alpha(x + x^\star) & \text{“antiboost”} \\
 & \\
x^\star = x & \\
t^\star = e^\delta t & \text{time dilatation}
\end{array} \tag{2.4}$$

"Antiboosts" are particularly interesting : $x^\star$ and $t^\star$ are only defined implicitely, and the action is "field–dependent" in that its very definition involves Θ.

When implemented on the fields non–conventionally, these transformations act as symmetries. In detail, let us set

$$
\begin{aligned}
&R^\star(x,t) = \frac{R(x^\star,t^\star)}{J^\star}, \qquad && \text{"antiboost"}\\
&\Theta^\star(x,t) = \Theta(x^\star,t^\star); && \\
& && \\
&R^\star(x,t) = e^{-\delta}R(x^\star,t^\star), && \text{time dilatation}\\
&\Theta^\star(x,t) = e^{\delta}\Theta(x^\star,t^\star); &&
\end{aligned}
\tag{2.5}
$$

where $J^\star = \left(1 - \alpha\partial_{x^\star}\Theta(x^\star,t^\star) - \frac{1}{2}\alpha^2\partial_{t^\star}\Theta(x^\star,t^\star)\right)^{-1}$ is the Jacobian of the space-time transformation (2.4). Then the Lagrangian (1.1) changes by a surface term and the conserved quantities (1.5) are recovered by Noether's theorem.

So far we merely reviewed the results from Ref. [3]. Now we point out that, in the free case $V = 0$, the system described by the Lagrangian $\mathcal{L}_0$ in (1.1) has even more symmetries. Let us first remember that the "non-relativistic conformal transformations"

$$
\begin{aligned}
&x^\star = e^{\lambda/2}x, \qquad && \text{non-relat. dilatations}\\
&t^\star = e^{\lambda}t; && \\
& && \\
&x^\star = \frac{x}{1-\kappa t}, && \text{expansions}\\
&t^\star = \frac{t}{1-\kappa t}; &&
\end{aligned}
\tag{2.6}
$$

are symmetries for the free Schrödinger equation [8]. The transformations in (2.6) span with the time translation $t^\star = t + \epsilon$ an SL(2,$\mathbb{R}$) group; added to the Galilei transformations (2.3), the Schrödinger group is obtained.

Implementing (2.6) on R and Θ as

$$\begin{aligned}
&\left.\begin{array}{l} R^\star(x,t) = e^{\lambda/2} R(x^\star, t^\star), \\ \Theta^\star(x,t) = \Theta(x^\star, t^\star); \end{array}\right. && \text{non-relat. dilatation} \\
&\left.\begin{array}{l} R^\star(x,t) = \dfrac{1}{1-\kappa t}\, R(x^\star, t^\star), \\ \Theta^\star(x,t) = \Theta(x^\star, t^\star) - \dfrac{\kappa x^2}{2(1-\kappa t)}; \end{array}\right. && \text{expansion}
\end{aligned} \tag{2.7}$$

the free action is left invariant. Thus, the transformations in (2.6) act as symmetries also in our case. The associated conserved quantities read

$$\begin{aligned}
&\Delta = \int dx\, (t\mathcal{H} - \frac{1}{2} x\mathcal{P}), && \text{non-relativistic dilatation} \\
&K = -t^2 H + 2t\Delta + \tfrac{1}{2} \int dx\, x^2 R; && \text{expansion}
\end{aligned} \tag{2.8}$$

Remarkably, the "relativistic" dynamical Poincaré symmetry can also be conformally extended. Using the equations of motion (1.2), a lengthy but straightforward calculation shows that, for $V = 0$,

$$\begin{aligned}
\mathrm{C}_1 &= \int dx\, \Big(\frac{x^2}{2}\mathcal{H} - x\Theta\mathcal{P} + \Theta^2 R\Big), \\
\mathrm{C}_2 &= \int dx \Big(xt\mathcal{H} - (\frac{x^2}{2} + t\Theta)\mathcal{P} + x\Theta R\Big)
\end{aligned} \tag{2.9}$$

are also conserved, $\frac{d\mathrm{C}_i}{dt} = 0$. A shorter proof can be obtained by calculating the energy–momentum tensor for (1.1),

$$\begin{aligned}
T_{tt} &= \frac{R}{2} (\partial_x \Theta)^2 + cR^\omega, \\
T_{xt} &= -R\partial_x\Theta\, \partial_t\Theta, \\
T_{tx} &= R\partial_x\Theta, \\
T_{xx} &= R(\partial_x\Theta)^2 + (\omega - 1)cR^\omega.
\end{aligned} \tag{2.10}$$

(Let us note that in the non–relativistic context the usual index gymnastics is meaningless, since space-time does not carry a metric. We agree therefore

that the energy–momentum tensor $T_{\alpha\beta}$ is a covariant two tensor carrying lower indices, while $T^{\alpha\beta}$ is not defined).

The tensor $T_{\alpha\beta}$ is neither symmetric nor traceless. It is nevertheless conserved, $\partial_\alpha T_{\alpha\beta} = 0$ for all $\beta = t, x$. Let us rewrite the quantities C_1 and C_2 as the integrals of

$$\begin{aligned} &\frac{x^2}{2}T^0_{tt} - x\Theta T^0_{tx} + \Theta^2 R && \mathrm{C}_1, \\ &xtT^0_{tt} - (\frac{x^2}{2} + t\Theta)T^0_{tx} + x\Theta R && \mathrm{C}_2, \end{aligned} \tag{2.11}$$

where $T^0_{\alpha\beta}$ denotes the free ($V = 0$) energy-momentum tensor. Then the conservation of the quantities (2.9) is obtained by deriving this expression w. r. t. time and using the continuity equation $\partial_\alpha T^0_{\alpha\beta} = 0$.

The Poisson brackets of our conserved quantities (listed in Appendix A) yield a closed, finite–dimensional algebra. In the next Section we prove that this is in fact the $\mathrm{o}(3,2)$ conformal algebra.

For the membrane potential $V = c/R$, the conformal symmetries Δ, K, C_1 and C_2 are broken, and only the Poincaré symmetry survives. Changing the question, we can also ask for what potentials do we have the same symmetries as in the free case. Now the Lagrangian (1.1) is dilation and indeed Schrödinger invariant only for

$$V = cR^3. \tag{2.12}$$

This comes from the scaling properties of the Lagrange density, and can also be seen of by looking at the energy-momentum tensor (2.10) : the trace condition

$$T_{xx} = 2T_{tt}, \tag{2.13}$$

which is the signal for a Schrödinger symmetry [8], only holds for $\omega = 3$. On the other hand, the potential cR^ω yields an "antiboost–invariant" expression only for $\omega = -1$ so that he Poincaré symmetry only allows the "membrane potential" (1.3), $V(R) = c/R$. Therefore, the full $\mathrm{o}(3,2)$ conformal symmetry only arises in the free case.

3 A "Kaluza-Klein" framework

In order to explain the origin of the symmetries of the model, let us start with Duval's unpublished observation [10]. Let us enlarge space-time by adding a new, "phase-like" coordinate s i.e., consider the "extended space"

$$M = \left\{ \begin{pmatrix} x \\ t \\ s \end{pmatrix} \right\}. \tag{3.1}$$

Let us lift the space–time transformations to M by adding a transformation rule for s inspired from the rule the phase changes in Eq. (2.5). Thus, let us formally replace the field $\Theta^\star(x,t)$ by the coordinate $-s$,

$$\Theta^\star(x,t) \to -s. \tag{3.2}$$

When applied to an "antiboost", for example, we get the *linear* action on extended space[1], $\begin{pmatrix} x \\ t \\ s \end{pmatrix} \to \begin{pmatrix} \widetilde{x} \\ \widetilde{t} \\ \widetilde{s} \end{pmatrix}$,

$$G \quad : \qquad \begin{array}{l} \widetilde{x} = x - \alpha s, \\ \widetilde{t} = t + \alpha x - \frac{1}{2}\alpha^2 s, \\ \widetilde{s} = s. \end{array} \tag{3.3}$$

On the other hand, lifting the action of galilean boosts to our extended space-time by applying the same rules, we get

$$B \quad : \qquad \begin{array}{l} \widetilde{x} = x + \beta t, \\ \widetilde{t} = t, \\ \widetilde{s} = s - \beta x - \frac{1}{2}\beta^2 t. \end{array} \tag{3.4}$$

The action of the mysterious "antiboost" becomes hence analogous to that of galilean boost, the only difference being that ordinary time, t, and the new, phase-like coordinate, s, have to be interchanged,

$$t \longleftrightarrow s. \tag{3.5}$$

[1] Our notations are as follows. $\mu, \nu, \ldots = x, t, s$ are indices on the extended space M, and $\alpha, \beta \ldots = x, t$ are indices on ordinary space–time, Q. The transformed coordinates are denoted by "tilde" $(\widetilde{\{\cdot\}})$ on M, while they are denoted by "star", $(\{\cdot\}^\star)$, on Q. The fields on M are denoted by lower–case letters (e. g., ρ, θ), while the fields on Q are denoted by upper–case letters like R, Θ.

When interchanging t and s, the dilations of time alone in Eq. (2.4) lifted to extended space by the same rule as above, remain dilations of time alone but with the inverse parameter : $\delta \to -\delta$,

$$D \ : \quad \begin{array}{l} \widetilde{x} = x, \\ \widetilde{t} = e^{\delta} t, \\ \widetilde{s} = e^{-\delta} s \end{array} \quad \Longrightarrow \quad \begin{array}{l} \widetilde{x} = x, \\ \widetilde{t} = e^{-\delta} t, \\ \widetilde{s} = e^{\delta} s. \end{array} \tag{3.6}$$

This same rule changes a time translation with parameter $\epsilon = -\eta$ into a the phase translation,

$$\begin{array}{l} \text{time translation} \\ \widetilde{x} = x \\ \widetilde{t} = t + \epsilon \\ \widetilde{s} = s \end{array} \qquad \Longrightarrow \qquad \begin{array}{l} \text{phase translation} \\ \widetilde{x} = x \\ \widetilde{t} = t \\ \widetilde{s} = s - \eta \end{array} \tag{3.7}$$

i.e.,

$$\text{energy} \qquad \Longrightarrow \qquad \text{particle number}$$

Our trick of adding an extra coordinate s allowed us so far to reconstruct the Poincaré group from the extended Galilei group by the “interchange rule” (3.5). The conformal extensions can be similarly investigated. Non–relativistic dilations act as

$$\Delta : \begin{pmatrix} x \\ t \\ s \end{pmatrix} \ \to \ \begin{pmatrix} e^{\lambda/2} x, \\ e^{\lambda} t; \\ s \end{pmatrix}. \tag{3.8}$$

Let us observe that relativistic dilations, i. e., uniform dilations of all coordinates, $d : \begin{pmatrix} x \\ t \\ s \end{pmatrix} \to \begin{pmatrix} e^{\delta} x \\ e^{\delta} t \\ e^{\delta} s \end{pmatrix}$ also belong to our algebra, since they correspond to a non-relativistic dilation (Δ) with parameter 2δ, followed by a dilation of time alone (D) with parameter $-\delta$, $d = D_{-\delta} \circ \Delta_{2\delta}$. Then the $t \leftrightarrow s$ counterpart of a non–relativistic dilation is a uniform dilation followed by a dilation of time alone,

$$\Delta_{\lambda} \to D_{-\lambda/2} \circ d_{\lambda/2}. \tag{3.9}$$

The $s \leftrightarrow t$ counterpart of non–relativistic expansions (2.6)–(2.7) with parameter $\kappa = -\epsilon_1$ is in turn a new transformation we denote by C_1,

$$\begin{array}{lcl}
\text{expansions} & & \mathrm{C}_1 \\[2mm]
\begin{aligned}
\widetilde{x} &= \frac{x}{1-\kappa t} \\
\widetilde{t} &= \frac{t}{1-\kappa t} \\
\widetilde{s} &= s - \frac{\kappa x^2}{2(1-\kappa t)}
\end{aligned}
& \Longrightarrow &
\begin{aligned}
\widetilde{x} &= \frac{x}{1+\epsilon_1 s} \\
\widetilde{t} &= t + \frac{\epsilon_1 x^2}{2(1+\epsilon_1 s)} \\
\widetilde{s} &= \frac{s}{1+\epsilon_1 s}
\end{aligned}
\end{array} . \tag{3.10}$$

The infinitesimal version of the new transformation is

$$X_8 = \frac{x^2}{2}\partial_t - xs\partial_x - s^2\partial_s. \tag{3.11}$$

Calculating the Lie brackets of (3.11) with the other infinitesimal transformations, we get one more vectorfield. In fact, the bracket of (3.11) with the generator of infinitesimal boosts, $t\partial_x - x\partial_s$, yields

$$X_9 = xt\partial_t + (\frac{x^2}{2} - ts)\partial_x + xs\partial_s. \tag{3.12}$$

Collecting our results, our symmetry generators read

$$\begin{array}{lcll}
X_0 & = & \partial_t & \text{time translation} \\
X_1 & = & -\partial_x & \text{space translation} \\
X_2 & = & -\partial_s & \text{vertical translation} \\
X_3 & = & t\partial_x - x\partial_s & \text{galilean boost} \\
X_4 & = & t\partial_t + \frac{x}{2}\partial_x & \text{non-relat. dilatation} \\
X_5 & = & t^2\partial_t + xt\partial_x - \frac{x^2}{2}\partial_s & \text{expansion} \\
X_6 & = & t\partial_t - s\partial_s & \text{time dilation} \\
X_7 & = & x\partial_t - s\partial_x & \text{“antiboost”} \\
X_8 & = & \frac{x^2}{2}\partial_t - xs\partial_x - s^2\partial_s & \mathrm{C}_1 \\
X_9 & = & xt\partial_t + (\frac{x^2}{2} - ts)\partial_x + xs\partial_s & \mathrm{C}_2
\end{array} \tag{3.13}$$

The Lie brackets of these vector fields are seen to satisfy the same algebra as the conserved quantities in (8.1). The vectorfields X_8 and X_9 will be shown below in particular to generate the two additional conserved quantites C_1 and

C_2 in Eq. (2.9). Note that the algebra (3.13) is manifestly invariant w. r. t. the interchange $t \longleftrightarrow s$. The vector field X_9 is itself invariant; this is the reason why we could not find it by the "interchange–trick".

The extended manifold M above has already been met before. In their "Kaluza-Klein-type" framework for non-relativistic physics in $d+1$ dimension, Duval et al. [9] indeed consider a $(d+1,1)$–dimensional Lorentz manifold $(M, g_{\mu\nu})$, endowed with a covariantly constant lightlike "vertical" vector $\xi = (\xi^\mu)$ they call "Bargmann space". The quotient of M by the flow of ξ is a non-relativistic space-time denoted by Q. In the application we have in mind, M is simply 3-dimensional Minkowski space, with the usual coordinates x_0, x, y and metric $-(dx^0)^2 + dx^2 + dy^2$. Introducing the light-cone coordinates

$$t = \frac{1}{\sqrt{2}}(y - x^0), \qquad s = \frac{1}{\sqrt{2}}(y + x^0), \tag{3.14}$$

the Minkowskian metric reads $dx^2 + 2dtds$. Then $\xi = \partial_s$ is indeed lightlike and covariantly constant.

All [infinitesimal] conformal transformations of Minkowski space form the conformal algebra $o(3,2)$. Now, as shown in Appendix A, the X_i found above provide just another basis of this same algebra.

4 Conformal geometry

The action of the orthogonal group $O(3,2)$ on 3-dimensional Minkowski space is the best described as follows. Consider the natural action of $O(3,2)$ on $\mathbb{R}^{3,2}$ by matrix multiplication. A vector in $\mathbb{R}^{3,2}$ can be written as

$$Y = \begin{pmatrix} y \\ a \\ b \end{pmatrix}, \qquad \text{where} \qquad y = \begin{pmatrix} x \\ t \\ s \end{pmatrix} \in \mathbb{R}^{2,1}, \ a,\ b \in \mathbb{R}. \tag{4.1}$$

The vector space $\mathbb{R}^{3,2}$ carries the quadratic form $\bar{Y}Y = \bar{y}y + 2ab$, where $\bar{y}y$ means $\bar{y}y = x^2 + 2ts$, so that $\bar{Y}$ is represented by the row-vector $(\bar{y}, b, a)$ where $\bar{y} = (x, s, t)$. $(2+1)$-dimensional Minkowski space, $M = \mathbb{R}^{2,1}$, can be mapped into the isotropic cone (quadric) $\mathcal{Q}$ in $\mathbb{R}^{3,2}$, as

$$y \mapsto \begin{pmatrix} y \\ 1 \\ -\frac{1}{2}\bar{y}y \end{pmatrix}. \tag{4.2}$$

Projecting onto the real projective space $P\mathcal{Q}$, we identify M with those generators in the null-cone in $\mathbb{R}^{3,2}$. The manifold $P\mathcal{Q}$ is invariant with respect to the action of $\mathrm{O}(3,2)$.

Let us first consider infinitesimal actions. An $\mathrm{o}(3,2)$ matrix can be written as

$$\begin{pmatrix} \Lambda & V & W \\ -\bar{W} & -\lambda & 0 \\ -\bar{V} & 0 & \lambda \end{pmatrix}, \qquad \Lambda \in \mathrm{o}(2,1),\, V,\, W \in \mathbb{R}^{2,1},\ \lambda \in \mathbb{R}. \tag{4.3}$$

The matrix action of $\mathrm{o}(3,2)$ on $\mathbb{R}^{3,2}$ yields the action on Bargmann space

$$\Lambda y + V - \tfrac{1}{2} W \bar{y} y + (\bar{W} y + \lambda) y. \tag{4.4}$$

In particular, V represents infinitesimal translations. Observe now that the covariantly constant null vector ξ is also the generator of vertical translations,

$$\hat{\xi} = \begin{pmatrix} 0 & \xi & 0 \\ 0 & 0 & 0 \\ -\bar{\xi} & 0 & 0 \end{pmatrix}. \tag{4.5}$$

The Schrödinger algebra is identified as those vectorfields which commute with the "vertical vector",

$$[Z, \hat{\xi}] = 0. \tag{4.6}$$

This yields the constraints $\Lambda\xi = -\lambda\xi$ and $W = \kappa\xi$, $\kappa \in \mathbb{R}$. It follows that

$$Z = \begin{pmatrix} 0 & \beta & 0 & \gamma & 0 \\ 0 & \lambda & 0 & \tau & 0 \\ -\beta & 0 & -\lambda & \eta & \kappa \\ 0 & -\kappa & 0 & -\lambda & 0 \\ -\gamma & -\eta & -\tau & 0 & \lambda \end{pmatrix}, \qquad \beta, \gamma, \lambda, \kappa, \eta \in \mathbb{R}. \tag{4.7}$$

This is the *extended Schrödinger algebra*, with

- β representing Galilei boosts,
- γ space translations,
- τ time translations,
- λ non-relativistic dilatations,
- κ expansions,
- η translations in the vertical direction.

Using (4.4), we recover the infinitesimal action of the (extended) Schrödinger algebra on M [9]. Note that the *relativistic* dilation invariance [with all directions dilated by the same factor], is *broken* by the reduction: only doubly time-dilated combinations project to Bargmann space.

Those parameterized by $\beta, \gamma, \tau, \eta \in \mathbb{R}$ are isometries and are recognized as the generators of the *extended Galilei* (or Bargmann) group.

Let us now identify the unusual generators. "Antiboosts" and dilatations of time alone belong to the upper–left $\mathrm{o}(2,1)$ corner Λ of the $\mathrm{o}(3,2)$ matrix, (4.3)

$$\Lambda = \begin{cases} \begin{pmatrix} 0 & 0 & -\alpha \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{antiboost} \\ \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & -d \end{pmatrix} & \text{dilation of time alone} \end{cases} \tag{4.8}$$

Augmented with the extended Galilei algebra, the Poincaré algebra is obtained.

In the same spirit, the two remaining (relativistic) conformal transformations C_1 and C_2 correspond to chosing $W_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, respectively. The generated group, found by exponentiating, is the conformal group $\mathrm{SO}(3,2)$.

The Schrödinger group is recovered as those transformations which commute with the 1-parameter subgroup generated by $\hat{\xi}$. It acts on M according to in the standard way which plainly project to "ordinary" space-time and span there the (non–extended) Schrödinger group consistently with (2.3) and

(2.6). The transformations which do *not* preserve ξ are

$$
\begin{array}{rclr}
\widetilde{x} &=& x & \\
\widetilde{t} &=& e^{\delta}t & \text{time dilation} \\
\widetilde{s} &=& e^{-\delta}s & \\[1em]
\widetilde{x} &=& x-\alpha s & \\
\widetilde{t} &=& t+\alpha x-\frac{1}{2}\alpha^2 s & \text{“antiboost”} \\
\widetilde{s} &=& s & \\[1em]
\widetilde{x} &=& \dfrac{x}{1+\epsilon_1 s} & \\
\widetilde{t} &=& t+\frac{1}{2}\dfrac{\epsilon_1 x^2}{1+\epsilon_1 s} & \mathrm{C}_1 \\
\widetilde{s} &=& \dfrac{s}{1+\epsilon_1 s} & \\[1em]
\widetilde{x} &=& \dfrac{x-\epsilon_2(\frac{1}{2}x^2+ts)}{(1-\frac{1}{2}\epsilon_2 x)^2+\frac{1}{2}\epsilon_2^2 ts} & \\
\widetilde{t} &=& \dfrac{t}{(1-\frac{1}{2}\epsilon_2 x)^2+\frac{1}{2}\epsilon_2^2 ts} & \mathrm{C}_2 \\
\widetilde{s} &=& \dfrac{s}{(1-\frac{1}{2}\epsilon_2 x)^2+\frac{1}{2}\epsilon_2^2 ts} &
\end{array}
\tag{4.9}
$$

Our transformations are indeed conformal since they satisfy $f^*g_{\mu\nu} = \Omega^2 g_{\mu\nu}$ see Appendix A. Note that the interchange $s \leftrightarrow t$ is also an isometry and carries the group $\mathrm{SO}(3,2)$ into another component of the conformal group $\mathrm{O}(3,2)$.

5 Projecting to ordinary space–time

As we said already, the quotient Q is 1+1-dimensional “ordinary” spacetime, labeled by x (position) and t (time). The projection $M \to Q$ means simply “forgetting” the vertical coordinate s : $\begin{pmatrix} x \\ t \\ s \end{pmatrix} \to \begin{pmatrix} x \\ t \end{pmatrix}$.

Next, we wish to relate the fields on extended and on ordinary space, respectively. Let us recall how this is done usually [9]. Let ψ denote a

complex field on M. Then, requiring the field to be equivariant,

$$\xi^\mu \partial_\mu \psi = i\psi, \tag{5.1}$$

allows us to reduce ψ from Bargmann space to one on ordinary space-time as $\Psi(x,t) = e^{-is}\psi(x,t,s)$ [9]. Writing $\psi = \rho^{1/2} e^{i\theta}$, (5.1) reads $\xi^\mu \partial_\mu \rho = 0$ and $\xi^\mu \partial_\mu \theta = 1$. In light–cone coordinates of Minkowski case in particular, these conditions imply that

$$R(x,t) = \rho(x,t,s), \qquad \Theta(x,t) = \theta(x,t,s) - s. \tag{5.2}$$

are well–defined fields on Q. These formulæ (also referred to as equivariance) allows us to relate equivariant fields on extended space to fields on ordinary space.

Let us now consider a diffeomorphism

$$f(x,t,s) \equiv \begin{pmatrix} \widetilde{x} \\ \widetilde{t} \\ \widetilde{s} \end{pmatrix} = \begin{pmatrix} g(x,t,s) \\ h(x,t,s) \\ k(x,t,s) \end{pmatrix} \tag{5.3}$$

of M. How can we project this to ordinary space–time ? In the particular case when the mapping preserves ξ, the entire fibre goes into the same fibre and the result projects to a well–defined diffeomorphism of ordinary space-time. In fact, $\widetilde{x} = g(x,t)$, $\widetilde{t} = h(x,t)$, $\widetilde{s} \equiv k(x,t,s) = s + K(x,t)$, so that we can define the projected map $F(x,t) = \begin{pmatrix} x^\star \\ t^\star \end{pmatrix}$ by setting $x^\star = \widetilde{x} = g(x,t)$ and $t^\star = \widetilde{t} = h(x,t)$. As a bonus, we also get the usual transformation rule of the phase (consistent with the equivariance), $\Theta^\star(x,t) = \Theta(x^\star, t^\star) + K(x,t)$.

If, however, f does not preserve the fibres, this construction does not work since the coordinates $\widetilde{x}$ and $\widetilde{t}$ now depend on s. Hence the need of generalizing the construction based on equivariance. Forgetting momentarily about ρ, we only consider the phase, θ. Our clue is to observe that if θ is equivariant, (5.2), then $s = -\Theta(x,t)$ is solution of the equation $\theta(x,t,s) = 0$, i. e.,

$$\theta(x,t,-\Theta(x,t)) = 0. \tag{5.4}$$

This condition is, however, meaningful without any assumption of equivariance and associates implicitly a function $\Theta(x,t)$ to each x, t *and* field θ.

Conversely, to any x, t and $\Theta(x,t)$ Eq. (5.2) associates an (equivariant) field $\theta(x,t,s)$ on M.

Let us recall that our extended "Bargmann" space M is a fibre bundle over ordinary space-time Q, with fibre $\mathbb{R}$. Then Θ corresponds to a section $Q \to M$ of this bundle. Condition (5.4) requires the existence of a section $\Theta(x,t)$ along which the phase field θ vanishes.

A diffeomorphism f of M acts naturally on θ, namely as

$$\widetilde{\theta} = f^\star \theta. \tag{5.5}$$

We can define therefore $\Theta^\star$ as the solution of the equation

$$\widetilde{\theta}(x,t,-\Theta^\star(x,t)) = 0. \tag{5.6}$$

This implicit equation (assumed to admit a unique solution) associates a $\Theta^\star$ to x, t and θ. Let us stress that $\widetilde{\theta}$ is not in general equivariant even if θ is equivariant, unless f preserves the fibres.

Thus, starting with $\Theta(x,t)$ we lift it first to M as an equivariant field $\theta(x,t,s) = \Theta(x,t) + s$ on M; to which a well–defined $\Theta^\star$ (function of x, t and Θ) is associated by (5.6). Having defined $\Theta^\star$, the diffeomorphism f of M can be projected to Q in a "field–dependent way" by restricting $\widetilde{f}$ to the section $s = -\Theta^\star$. In coordinates, $F(x,t) = \begin{pmatrix} x^\star \\ t^\star \end{pmatrix}$, where

$$\begin{aligned} x^\star &= g(x,t,-\Theta^\star(x,t)), \\ t^\star &= h(x,t,-\Theta^\star(x,t)), \\ \Theta(x^\star,t^\star) &= -k(x,t,-\Theta^\star(x,t)). \end{aligned} \tag{5.7}$$

The last line here requires to express $\Theta^\star$ by inverting the function k and reinserting the result into the two first lines. It also implements the transformation on the "phase", Θ. Let us stress that these formulæ are implicit : $x^\star$ and $t^\star$ can not be defined without defining $\Theta^\star$, which itself involves $x^\star$ and $t^\star$.

In the equivariant case, the procedure is plainly consistent with the previous formulae.

In the non-fiber-preserving case it yields the "field–dependent diffeomorphisms" considered by Bazeia and Jackiw [3]. For "antiboosts", for example,

we get from (3.3)

$$\begin{aligned} x^\star &= x + \alpha\Theta^\star(x,t), \\ t^\star &= t + \alpha x + \tfrac{1}{2}\alpha^2\Theta^\star(x,t) \\ \Theta^\star(x,t) &= \Theta(x^\star, t^\star), \end{aligned} \tag{5.8}$$

which is equivalent to the definition (2.4). Time dilations work similarly. The formulae valid for the two relativistic conformal transformations, C_1 and C_2 above, is presented in Appendix B (9.1). The formula for C_1 is consistent with (3.10); that for C_2 is a new result.

So far, we only studied how to act on Θ : (5.7) only involves the phase but not the density. Turning to this problem, let us *posit*

$$R(x,t) = \rho(x,t,-\Theta(x,t))\partial_s\theta(x,t,-\Theta(x,t)), \tag{5.9}$$

where Θ is defined by (5.4). $R(x,t)$ is a well–defined function of x and t. Let us insist that (5.9) is again "field–dependent" in that it also depends on θ, except when θ is equivariant, when it reduces to (5.2). Conversely, if $R(x,t)$ is any field on Q, $\rho(x,t,s) = R(x,t)$ can obviously be viewed as (an equivariant) function on extended space.

Let us henceforth consider a conformal transformation f of M $f^\star g_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and let ρ be a (possibly not equivariant) field on M. f acts naturally on ρ as

$$\rho \to \widetilde{\rho} = \Omega f^* \rho. \tag{5.10}$$

Hence

$$R^\star(x,t) = \widetilde{\rho}(x,t,-\Theta^\star(x,t))\partial_s\widetilde{\theta}(x,t,-\Theta^\star(x,t)). \tag{5.11}$$

Using the definition (5.9) of R, this is also written as

$$R^\star(x,t) = \Omega(x,t,-\Theta^\star)\,\frac{\partial_s\widetilde{\theta}(x,t,-\Theta^\star(x,t))}{\partial_s\theta(x^\star,t^\star,-\Theta(x^\star,t^\star))}R(x^\star,t^\star). \tag{5.12}$$

(If the field θ is equivariant, the denominator is equal to 1). On the other hand, one can show in general that

$$\frac{\partial_s\widetilde{\theta}(x,t,-\Theta^\star(x,t))}{\partial_s\theta(x^\star,t^\star,-\Theta(x^\star,t^\star))} = \frac{\widetilde{J}(x,t,-\Theta^\star(x,t))}{J^\star(x,t)}, \tag{5.13}$$

where J^* and $\widetilde{J}$ are the Jacobians on ordinary and on the extended space respectively,

$$J^\star = \det\left(\frac{\partial (x^\star)^\alpha}{\partial x^\beta}\right) \qquad \widetilde{J} = \det\left(\frac{\partial \widetilde{x}^\mu}{\partial x^\nu}\right), \tag{5.14}$$

($\alpha, \beta = x, t$ and $\mu, \nu = x, t, s$). Eq. (5.12) can therefore be rewritten as

$$R^\star(x,t) = \Omega(x,t,-\Theta^\star(x,t)) \times \frac{\widetilde{J}(x,t,-\Theta^\star(x,t))}{J^\star(x,t)}\, R(x^\star,t^\star). \tag{5.15}$$

For a conformal transformation $\widetilde{J} = \pm\Omega^3$, the sign depending on the mapping being orientation–preserving or not.

If the transformation f preserves ξ, Ω is a function of t–alone cf. (8.4) in Appendix A. Then $\widetilde{\theta}$ is again equivariant and our formula reduces to the standard expression

$$R^\star(x,t) = \Omega(t)\, R(x^\star,t^\star), \tag{5.16}$$

cf. (2.7). For an isometry, $\Omega = 1$, so that (5.15) reduces to

$$R^\star(x,t) = \frac{R(x^\star,t^\star)}{J^\star(x,t)}. \tag{5.17}$$

For time dilations and "antiboosts", the formulæ of Bazeia and Jackiw in [3], (our (1.5)), are recovered. For the relativistic conformal transformations C_1 and C_2, we find some complicated expressions (9.2), presented in Appendix B.

Our formulae allow to implement any isometry of M, not only those in the connected component of the Poincaré group. Let us consider, for example, the interchange

$$t \longleftrightarrow s, \tag{5.18}$$

which is a non–fiber-preserving isometry. It acts on the fields defined on M in the natural way. For fields on Q, we get the "field-dependent action"

$$\begin{aligned} x^\star &= x, \\ t^\star &= -\Theta^\star(x,t), \\ \Theta(x,-\Theta^\star) + t &= 0, \\ R^\star(x,t) = R(x,-\Theta^\star)\partial_t\Theta(x,-\Theta^\star) &= \frac{R(x,-\Theta^\star)}{\partial_t\Theta^\star(x,t)}. \end{aligned} \tag{5.19}$$

This formula is so much implicit that we can not go farther unless Θ is given explicitly. It is nevertheless a "field–dependent symmetry".

The weak condition (5.6) can hence accomodate the $t \leftrightarrow s$ symmetry. This is in sharp contrast with the equivariance condition (5.1) which manifestly breaks it. Note also that our formulæ for implementing the conformal transformations on the fields are consistent with the interchange symmetry $t \leftrightarrow s$, followed by the rule of replacing s with $-\Theta^{\star}$. When applied to a Schrödinger transformation, it yields its non–ξ–preserving counterpart.

6 Physics on extended space

So far the "Bargmann space" M was only used as a geometric arena for linearizing the action of the conformal group. Now we show how to lift the *physics* to M. Generalising our previous theory, let M be $(d+1,1)$–dimensional Lorentz manifold $(M, g_{\mu\nu})$ endowed with a covariantly constant lightlike vector $\xi = (\xi^{\mu})$. Such a manifold admits a preferred coordinates $\vec{x}, t, s$ in which the metric is

$$g_{ij}(\vec{x},t)dx^i dx^j + 2dt[ds + \vec{A}\cdot d\vec{x}] - 2U(\vec{x},t)dt^2, \tag{6.1}$$

where g_{ij} is a metric on d–dimensional "transverse space" and $\vec{A}$ and U are vector and scalar potentials, respectively [12], [9].

6.1 Field theory on extended space

Let ρ and θ be two real fields on M, and let us consider the field theory described by the action

$$\begin{aligned} S = \quad & S_0 + S_p = \\ & \int -\frac{1}{2}\left(\rho\nabla_\mu\theta\nabla^\mu\theta\right)\sqrt{-g}\,d^3x - \int V(\rho)\sqrt{-g}\,d^3x, \end{aligned} \tag{6.2}$$

where ∇_μ is the covariant derivative associated with the metric of M. The Euler-Lagrange equations read

$$\nabla_\mu(\rho\nabla^\mu\theta) = 0, \qquad \tfrac{1}{2}\nabla_\mu\theta\nabla^\mu\theta = -\frac{dV}{d\rho}. \tag{6.3}$$

When the fields are required to be also equivariant, (5.2), then, for the projected variables Θ and ρ, the equations of motion (6.3) reduce to those

of Bazeia and Jackiw in Ref. [3], Eqn. (1.2) above. (Working with a general Bargmann space [9] would allow us to describe our fluid system in an external electromagnetic field).

Equivariance is a too strong condition, though. For specific potentials, the weaker conditions (5.4) and (5.12), i.e.

$$\begin{aligned} &\theta(x,t,-\Theta(x,t)) = 0, \\ &R(x,t) = \rho(x,t,-\Theta(x,t))\partial_s\theta(x,t,-\Theta(x,t)) \end{aligned} \tag{6.4}$$

may still work. Expressing $\partial_\alpha\Theta$ by deriving the defining relation (5.4) one finds using the Euler–Lagrange equations (6.3) that R and Θ satisfy the Bazeia–Jackiw (1.2) *provided* $V(\rho)$ is the membrane potential $V(\rho) = c/\rho$. This is hence the only potential consistent with (5.4).

In sharp contrast with equivariance, our new condition does not impose any restriction to the fields. Let us consider, for example, the action (6.2) on $(2+1)$ dimensional Minkowski space with $V = 0$ and chose

$$\rho = \sqrt{R} \qquad \text{and} \qquad \theta = \sqrt{R}\,\sin(\Theta + s). \tag{6.5}$$

The corresponding field on Bargmann space, $\psi = R^{1/4}e^{i\sqrt{R}\,\sin(\Theta+s)}$, is not equivariant. This Ansatz satisfies however our conditions (6.4), as anticipated by the notations. It projects to (1.1) with its large symmetry.

6.2 Symmetries

The Kaluza-Klein type framework is particularly convenient for studying the symmetries. Let us indeed consider a conformal diffeomorphism $f(x,t,s)$ of the Bargmann metric. It is easy to see, along the lines indicated in Refs. [9], that implementing f on the fields as

$$\begin{aligned} &\theta \to \widetilde{\theta} = f^*\theta, \\ &\rho \to \widetilde{\rho} = \Omega f^*\rho, \end{aligned} \tag{6.6}$$

the "free" action (6.2) is left invariant by all conformal transformations of M. This is explained by the absence of any mass term in (6.2). Equivalently, the transformed fields are seen to satisfy the equations of motion

$$\nabla_\mu(\widetilde{\rho}\,\nabla^\mu\widetilde{\theta}) = 0, \qquad \nabla_\mu\widetilde{\theta}\,\nabla^\mu\widetilde{\theta} = 0. \tag{6.7}$$

It is worth noting that unfolding to extended space converted the up–to-surface–term invariant system (1.1) into a strictly invariant one.

We can now derive once again the symmetries starting from the extended space. Let us first consider the free case. Differentiating the defining relations (5.6) and (5.11) we find, using the equations of motion (6.7) on M, that $R^\star$ and $\Theta^\star$ satisfy the free equations of motion in ordinary space,

$$\partial_t R^\star + \partial_x (R^\star \partial_x \Theta^\star) = 0, \qquad \partial_t \Theta^\star + \frac{1}{2}(\partial_x \Theta^\star)^2 = 0.$$

Alternatively, we check readily that

$$dxdtR^\star(x,t)\Big[\partial_t\Theta^\star(x,t) + \frac{1}{2}(\partial_x\Theta^\star(x,t))^2\Big] =$$
$$dx^\star dt^\star R(x^\star,t^\star)\Big[\partial_{t^\star}\Theta(x^\star,t^\star) + \frac{1}{2}(\partial_{x^\star}\Theta(x^\star,t^\star))^2\Big].$$

The free action (1.1) is hence invariant : each conformal transformation of extended space projects to a symmetry of the free system. Restoring the potential term, the scaling properties imply again that conformal symmetry on M only allows $V = c\rho^3$, cf (2.12). This potential is, however, inconsistent with the generalized condition (5.4) unless $c = 0$. Then we have the choice : if we keep $V = c\rho^3$ and use the usual equivariance (5.1), then the non–fiber preserving part is broken and we are left with a Schrödinger symmetry. If we choose $V = c/\rho$ the conformal symmetry is broken to its Poincaré subgroup from the outset; this survives, however, the reduction based on the generalised condition (5.4). In particular, the interchange $t \leftrightarrow s$, implemented as in (6.6) on θ and ρ (or on Θ and R as in (5.19)) is a symmetry.

6.3 Conserved quantities

On Bargmann space, we have a relativistic theory. Defining the energy–momentum tensor as the variational derivative of the action w. r. t. the metric, $\mathcal{T}_{\mu\nu} = 2\delta S/\delta g^{\mu\nu}$, we find

$$\mathcal{T}_{\mu\nu} = -\rho\,\nabla_\mu\theta\nabla_\nu\theta + \frac{\rho}{2}\,g_{\mu\nu}\nabla_\sigma\theta\nabla^\sigma\theta + g_{\mu\nu}V(\rho). \tag{6.8}$$

This energy–momentum tensor is symmetric, $\mathcal{T}_{\mu\nu} = \mathcal{T}_{\nu\mu}$, by construction and also manifestly. Using the equation of motion (6.3), we see at once that $\mathcal{T}_{\mu\nu}$ is traceless, $\mathcal{T}^{\mu}_{\ \mu} = 0$, precisely when $V = c\rho^3$ i.e., when our theory has the conformal symmetry. Finally, $\mathcal{T}_{\mu\nu}$ is conserved,

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 0, \tag{6.9}$$

as it follows from general covariance (i. e. from covariance w. r. t. diffeomorphisms [14]), and also from the eqns. of motion.

Let us assume that the potential is $V(\rho) = c\rho^3$ so that the system has conformal symmetry. To any conformal vector field $X = (X^\mu)$ on M, $L_X g_{\mu\nu} = \lambda g_{\mu\nu}$, we can now associate a conserved current [13] on M by contracting the energy-momentum tensor

$$k^\mu = \mathcal{T}^\mu_\nu X^\nu. \tag{6.10}$$

In fact, $\nabla_\mu k^\mu = (\nabla_\mu \mathcal{T}^\mu_\nu) X^\nu + \frac{1}{2} L_X g_{\mu\nu} \mathcal{T}^{\mu\nu} = 0$. The first term here vanishes beacause $\mathcal{T}_{\mu\nu}$ is conserved, and the second term vanishes because $\mathcal{T}_{\mu\nu}$ is traceless.

Let us assume henceforth that the fields ρ and θ are also equivariant. Then the Bargmann–space energy–momentum tensor $\mathcal{T}_{\mu\nu}$ becomes s-independent.

If X^μ commutes with the vertical vector ξ^μ, one can construct a conserved current on ordinary space out of k^μ as follows [13]. k^μ does not depend on s and projects therefore into a well-defined current J^α on Q, $(k^\mu) = (J^\alpha, k^s)$. The projected current is furthermore conserved, $\nabla_\alpha J^\alpha = 0$, because $\xi = \nabla_s$ is covariantly constant so that $\nabla_s k^s = 0$. In the general case, however, the current k^μ can not be projected in ordinary space, because it may depend on s; $\nabla_s k^s$ may also be non-vanishing.

Our idea is to construct a new current out of k^μ which does have the required properties. Let us restrict in fact k^μ to a "section" $s = -\Theta(x,t)$, i.e., define the Bargmann–space vector

$$j^\mu(x,t) = k^\mu(x,t,-\Theta(x,t)) = \Big(\mathcal{T}^\mu_\nu X^\nu\Big)(x,t,-\Theta(x,t)). \tag{6.11}$$

Then

$$\nabla_\mu j^\mu = \nabla_\alpha k^\alpha - \nabla_\alpha \Theta \nabla_s k^\alpha = -\nabla_s\Big(\nabla_x \Theta\, k^x + \nabla_t \Theta\, k^t + k^s\Big). \tag{6.12}$$

Inserting here the explicit form of k^μ we find that the bracketed quantity vanishes due to the equations of motion. The current j^μ is therefore conserved on M, $\nabla_\mu j^\mu = 0$. Let us now define the projected current as

$$J^\alpha(x,t) = \frac{j^\alpha(x,t)}{\nabla_s \theta(x,t,-\Theta(x,t))}. \tag{6.13}$$

It can shown using the equations of motion that J^α is a conserved current on Q, $\nabla_\alpha J^\alpha = 0$. Integrating the time–component of the projected current on ordinary space,

$$\int dx J^t \equiv \int dx \frac{\mathcal{T}_{\mu\nu}}{\nabla_s\theta} X^\mu \xi^\nu \equiv \int dx \frac{\mathcal{T}_{\mu s}}{\nabla_s\theta} X^\mu, \tag{6.14}$$

is hence conserved for any conformal vector $X = (X^\mu)$.

This yields the same conserved quantities as found before. The Bargmann-space energy–momentum tensor is in fact related to that in ordinary space, (2.10), according to

$$\begin{array}{lllll} (\nabla_s\theta)\, T_{tt} & = & -\mathcal{T}_t^t & = & -\mathcal{T}_{st}, \\ (\nabla_s\theta)\, T_{tx} & = & \mathcal{T}_x^t & = & \mathcal{T}_{sx}, \\ (\nabla_s\theta)\, T_{xt} & = & -\mathcal{T}_t^x & = & -\mathcal{T}_{xt}, \\ (\nabla_s\theta)\, T_{xx} & = & \mathcal{T}_x^x & = & \mathcal{T}_{xx}. \end{array} \tag{6.15}$$

Owing to the extra dimension, the Bargmann–space energy–momentum tensor admits the new component $\mathcal{T}_{ss}$ which, when contracted with the "vertical" component X^s of the lifted vector field, yields the $-C^0$ term in Noether's theorem (2.2). The situation is nicely illustrated by formulae like (2.11) of Section 2.

When X is fiber–preserving, we recover the generators H, P, B, N, Δ, K in (1.4) and (2.8) of the Schrödinger algebra.

For the non–fiber–preserving vectors, we get instead the new conserved quantities $G, D, \mathrm{C}_1, \mathrm{C}_2$ in (1.5) and (2.9).

Interchange, $t \leftrightarrow s$, acts on the Lie algebra of conserved quantities [10]. It carries in particular the energy to particle density, boosts to "antiboosts", etc., as already noted in Section 3.

7 The symmetries of the Schrödinger equation

We discuss now the (non–linear) Schrödinger equation in d spatial dimensions,

$$i\partial_t \Psi = -\frac{1}{2} \triangle \Psi - \frac{\partial \overline{V}(|\Psi|^2)}{\partial \Psi^*}. \tag{7.1}$$

where $\triangle$ is the d-dimensional Laplacian. When the wave function is decomposed into module, R, and phase, Θ, $\Psi = R^{1/2}e^{i\Theta}$, Eqn. (7.1) becomes indeed (1.2), with

$$V = \overline{V} + \frac{1}{8}\frac{(\partial_i R)^2}{R}. \tag{7.2}$$

A non-vanishing effective potential V is obtained therefore even for the *linear* Schrödinger equation $\overline{V} = 0$. The "free" theory described by $\mathcal{L}_0$ in Eqn. (1.1) corresponds hence to a non–linear Schrödinger equation (7.1) with effective potential $\overline{V} = -\frac{1}{8}\frac{\nabla_i R \nabla^i R}{R}$, this latter canceling the term coming from the hydrodynamical transcription. As we show below, canceling this effective term plays a crucial role.

Let us explain everything from the "Kaluza-Klein type" viewpoint. Generalizing to curved space, let us consider a complex scalar field ψ on a $d+2$ dimensional "Brinkmann" space M (6.1). Generalizing the flat–space results, we posit the action

$$S = \int \frac{1}{2}\nabla_\mu\psi\,\nabla^\mu\bar{\psi}\,\sqrt{-g}\,d^{d+2}x, \tag{7.3}$$

where $g = \det(g_{\mu\nu})$. The associated field equation is the curved–space massless Klein-Gordon (i.e., the free wave) equation

$$\nabla_\mu\nabla^\mu\psi = 0. \tag{7.4}$$

Equation (7.4) is *not* in general invariant w. r. t. conformal transformations of M, $f^\star g_{\mu\nu} = \Omega^2 g_{\mu\nu}$, implemented as

$$\psi \to \widetilde{\psi} = \Omega^{d/2} f^*\psi. \tag{7.5}$$

We explain this in the hydrodynamical transcription. Decomposing ψ as $\psi = \sqrt{\rho}\,e^{i\theta}$, the action (7.3) becomes

$$S = \int \left(\frac{1}{2}\rho\nabla_\mu\theta\nabla^\mu\theta + \frac{1}{8}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho}\right)\sqrt{-g}\,d^{d+2}x. \tag{7.6}$$

The action on the fields is now (6.6) i.e. $\theta \to \widetilde{\theta} = f^*\theta$, $\rho \to \widetilde{\rho} = \Omega^d f^*\rho$. As we have seen before, the first ("kinetic") term in (7.6) is invariant. The

second term is *not* invariant. Let us, however, modify the Lagrangian by adding a term which involves the scalar curvature $\mathcal{R}$ of M,

$$S_{\mathcal{R}} = \int \underbrace{\Big[\frac{1}{2}\nabla_\mu\psi\nabla^\mu\bar{\psi} + \frac{d}{8(d+1)}\mathcal{R}|\psi|^2\Big]}_{\mathcal{L}_R} \sqrt{-g}\, d^{d+2}x. \tag{7.7}$$

Then the symmetry-breaking terms will be absorbed by those which come from transforming $\mathcal{R}$, leaving a mere surface term (see Appendix C). In conclusion, the conformal symmetry *on* M is restored by the inclusion of the scalar curvature term as in Eq. (7.7), see [15]. Let us stress that this curvature term is only necessary due to the presence of the non-linear potential $\frac{1}{8}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho}$. Restoring the potential, the conformally invariant action is

$$S_{\bar{V}} = \int \left[\frac{1}{2}\nabla_\mu\psi\,\nabla^\mu\bar{\psi} + \frac{d}{8(d+1)}\mathcal{R}|\psi|^2 - \overline{V}(\psi^\star\psi)\right] \sqrt{-g}\, d^{d+2}x. \tag{7.8}$$

In Minkowski space $\mathcal{R} \equiv 0$. The curvature- term must nevertheless be added to the Lagrange density, since the conformally–transformed metric has already $\mathcal{R} \neq 0$. The scaling properties of the Lagrangian imply furthermore that $\overline{V}(\rho) = c\rho^{1+2/d}$ is the only potential consistent with the conformal symmetry $\mathrm{O}(d+2,2)$.

So far, we have only considered what happens on extended space. When the theory is reduced to ordinary space-time, some of the symmetries will be lost, however. We explain this when M is $(2+1)$–dimensional Minkowski space and for the linear Schrödinger equation $\overline{V} = 0$.

Firstly, the full conformal group (or its Poincaré subgroup) can only be projected to a (field–dependent) action on ordinary space-time using (5.4) and (5.6). However, the extended–space model only reduces to one of the Bazeia-Jackiw form (1.1) on Q when the potential is $V(\rho) = c/\rho$. The effective potential in (7.6) is manifestly not of this form, though. The weak condition (5.4) is hence inconsistent with the Schrödinger equation and has therefore to be discarded.

Under the assumption of equivariance instead, Eq. (5.2), the wave equation (7.4) on Minkowski space reduces, for $\Psi(x,t) = e^{-is}\psi(x,t,s)$, to the free Schrödinger equation

$$i\partial_t\Psi + \frac{1}{2}\partial_x^2\Psi = 0. \tag{7.9}$$

In terms of $R(x,t)$ and Θ where $\Psi = \sqrt{R}e^{i\Theta}$, this equation becomes

$$\begin{array}{l} \partial_t R + \partial_x(R\partial_x\Theta) = 0, \\ \partial_t\Theta + \frac{1}{2}(\partial_x\Theta)^2 = -\frac{1}{8}\frac{(\partial_x R)^2}{R^2} + \frac{\partial_x^2 R}{4R}. \end{array} \tag{7.10}$$

(Eqn. (7.10) does not contradict (1.2), since now $V = V(R, \partial_x R)$).

As explained in Section 5, usual equivariance only allows the Schrödinger subgroup to project : the "truly relativistic" generators G and D, (i.e., the antiboosts and the time dilations) as well as conformal generators C_1 and C_2 are hence broken by the reduction, leaving us with the mere Schrödinger symmetry [8], [9]. This latter is furthermore consistent with the potential $\overline{V}(\rho) = c\rho^{1+2/d}$.

The conserved quantities can be determined as indicated above. For the linear Schrödinger equation in $(1+1)$ dimensions, for example, the conserved energy-momentum tensor (10.1) in Appendix C allows to calculate the conserved quantities.

On extended space all conformal transformations are symmetries, and (6.10) associates a conserved current $k^\mu(,x,t,s)$, on M, $\nabla_\mu k^\mu = 0$, to each conformal generator. Its restriction to the section $s = -\Theta(x,t)$, $j^\mu(x,t)$ in (6.11), is not in general conserved, though. In $2+1$-dimensional Minkowski space, for example, the ξ–preserving transformations do yield conserved currents, namely the usual Schrödinger conserved quantities [8], [9]. However, the currents associated to ξ–non–preserving transformations as antiboosts, etc. are manifestly not conserved, as seen from (6.12).

Let us conclude our investigations with explaining how the results of Jevicki [2] fit into our framework. Let us start with the free wave equation (7.4) in $(2+1)$ dimensional Minkowski space and let us assume that the scalar field has the form

$$\psi = \frac{1}{4\pi}[\Psi(x,t)e^{is} + \Psi^\dagger(x,t)e^{-is}] = \frac{1}{2\pi}\sqrt{R(x,t)}\cos(\Theta + s), \tag{7.11}$$

where $\Psi(x,t) = \sqrt{R(x,t)}e^{i\Theta(x,t)}$. This field is not equivariant but is rather a mixture of two states with "masses" $(+1)$ and (-1). Hence the usual theory of [9] does not apply. Nor does it fit perfectly into our "weaker" theory : the phase is *identically* zero, so that any Θ solves our equation (5.4). Calculating

the Lagrange density for the Ansatz (7.11), we find, however,

$$\begin{aligned} -2\pi\mathcal{L}_0 = \quad & \Big\{\tfrac{1}{2}R(\partial_x\Theta)^2 + R\partial_t\Theta\Big\}\sin^2(\Theta+s) + \frac{(\partial_x R)^2}{8R}\cos^2(\Theta+s) \\ & -\Big\{R\partial_x\Theta + \partial_t R\Big\}\sin(\Theta+s)\cos(\Theta+s). \end{aligned} \tag{7.12}$$

The vertical direction can be compactified with period 2π. Then integrating over s yields the reduced action on ordinary space–time

$$-\int dxdt\left[\Big\{\tfrac{1}{2}R(\partial_x\Theta)^2 + R\partial_t\Theta\Big\} + \frac{(\partial_x R)^2}{8R}\right]. \tag{7.13}$$

Removing the effective potential $\frac{(\partial_x R)^2}{8R}$, we end up with the expression in (1.1). It has therefore the same $\mathrm{O}(3,2)$ conformal symmetry.

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References

[1] M. Bordemann and J. Hoppe, *The dynamics of relativistic membranes. Reduction to 2 dimensional fluid dynamics*. *Phys. Lett.* **B317**, 315 (1993);

[2] A. Jevicki, *Light-front partons and dimensional reduction in relativistic field theory*. *Phys. Rev.* **D57**, R5955 (1998).

[3] D. Bazeia and R. Jackiw, *Field-dependent diffeomorphism symmetry in diverse dynamical systems*. *Ann. Phys.* (N. Y.) **270**, 246 (1998).

[4] R. Jackiw and A. Polychronakos, *Dynamical Poincaré Symmetry Realized by Field-dependent Diffeomorphisms*. Faddeev *Festschrift*, Steklov Institute Proceedings. hep-th/9809123.

[5] D. Bazeia, *Galileo invariant system and the motion of relativistic d-branes. Phys. Rev.* **D 59**, 085007 (1999).

[6] R. Jackiw and A. Polychronakos, *Fluid dynamical profiles and constants of motion from d-branes.* To appear in *Comm. Math. Phys.*

[7] Y. Nutku, *J. Math. Phys.* **28**, 2579 (1987); P. Olver and Y. Nutku, *ibid.* **29**, 1610 (1988); M. Arik, F. Neyzi, Y. Nutku, P. Olver and J. Verosky, *ibid.* **30**, 1338 (1989).

[8] R. Jackiw, *Scaling symmetries. Physics Today* **25**, 23 (1980); U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation. Helvetica Physica Acta* **45**, 802 (1972); C. R. Hagen, *Scale and conformal transformations in Galilean-covariant field theory. Phys. Rev.* **D5**, 377 (1972).

[9] C. Duval, G. Burdet, H-P. Künzle and M. Perrin, *Bargmann structures and Newton-Cartan theory.* Phys. Rev. **D31**, 1841 (1985); C. Duval, G. Gibbons and P. Horváthy, *Celestial mechanics, conformal structures, and gravitational waves. Phys. Rev.* **D43**, 3907 (1991).

[10] C. Duval, unpublished notes (1998).

[11] R. Jackiw and N. Manton, *Symmetries and Conservation Laws in Gauge Theories. Annals of Physics* (N. Y.) **127**, 257 (1980). See also R. Jackiw, *Invariance, symmetry and periodicity in gauge theories. Acta Physica Austr.* (Suppl). **22**, 383 (1980).

[12] M. W. Brinkmann, *Enstein spaces which are mapped conformally on each other. Math. Ann.* **94**, 119 (1925).

[13] C. Duval, P. A. Horváthy and L. Palla, *Conformal symmetry of the coupled Chern–Simons and gauged nonlinear Schrödinger equations. Phys. Lett.* **B325**, 39 (1994).

[14] J.-M. Souriau, *Modèle de particule à spin dans le champ électromagnétique et gravitationnel. Ann. Inst. H. Poincaré* **20A**, 315 (1974).

[15] C. Callan, S. Coleman and R. Jackiw, *A new, improved energy-momentum tensor. Ann. Phys.* (N. Y.) **59**, 42 (1970).

[16] R. M. Wald, *General Relativity*. The University of Chicago Press: Chicago and London, (1984).

8 Appendix A : Lie algebra structure

The conserved quantities in (1.4), (1.5), (2.8) and (2.9) form a closed algebra. The Poisson brackets, $\{M,N\} = \int \left(\frac{\delta M}{\delta R}\frac{\delta N}{\delta \Theta} - \frac{\delta M}{\delta \Theta}\frac{\delta N}{\delta R} \right) dx$, read

$$
\begin{array}{lll}
\{H,P\} = 0, & \{H,N\} = 0, & \{H,B\} = P, \\
\{H,\Delta\} = H, & \{H,K\} = 2\Delta, & \{H,D\} = H, \\
\{H,G\} = 0, & \{H,\mathrm{C}_1\} = 0, & \{H,\mathrm{C}_2\} = G, \\
\{P,N\} = 0, & \{P,B\} = -N, & \{P,\Delta\} = \tfrac{1}{2}P, \\
\{P,K\} = B, & \{P,D\} = 0, & \{P,G\} = H, \\
\{P,\mathrm{C}_1\} = G, & \{P,\mathrm{C}_2\} = 2\Delta - D, & \{N,B\} = 0, \\
\{N,\Delta\} = 0, & \{N,K\} = 0, & \{N,D\} = -N, \\
\{N,G\} = -P, & \{N,\mathrm{C}_1\} = 2(D-\Delta), & \{N,\mathrm{C}_2\} = -B, \\
\{B,\Delta\} = -\tfrac{1}{2}B, & \{B,K\} = 0, & \{B,D\} = -B, \\
\{B,G\} = -D, & \{B,\mathrm{C}_1\} = \mathrm{C}_2, & \{B,\mathrm{C}_2\} = -K, \\
\{\Delta,K\} = K, & \{\Delta,D\} = 0, & \{\Delta,G\} = -\tfrac{1}{2}G, \\
\{\Delta,\mathrm{C}_1\} = 0, & \{\Delta,\mathrm{C}_2\} = \tfrac{1}{2}\mathrm{C}_2 & \{K,D\} = -K, \\
\{K,G\} = -\mathrm{C}_2, & \{K,\mathrm{C}_1\} = 0, & \{K,\mathrm{C}_2\} = 0, \\
\{D,G\} = -G, & \{D,\mathrm{C}_1\} = -\mathrm{C}_1, & \{D,\mathrm{C}_2\} = 0, \\
\{G,\mathrm{C}_1\} = 0, & \{G,\mathrm{C}_2\} = \mathrm{C}_1, & \{\mathrm{C}_1,\mathrm{C}_2\} = 0.
\end{array}
\tag{8.1}
$$

In light-cone coordinates, the generators of $\mathrm{o}(3,2)$ acting on Minkowski space are

$$
\begin{array}{ll}
\left.\begin{array}{l}
P_x = \partial_x \\
P_0 = \frac{1}{\sqrt{2}}(-\partial_t + \partial_s) \\
P_y = \frac{1}{\sqrt{2}}(\partial_t + \partial_s)
\end{array}\right\} & \text{translations} \\
\left.\begin{array}{l}
M_{01} = \frac{1}{\sqrt{2}}[(t-s)\partial_x - x(-\partial_t + \partial_s)] \\
M_{02} = -s\partial_s + t\partial_t \\
M_{12} = \frac{1}{\sqrt{2}}[-(t+s)\partial_x + x(\partial_t + \partial_s)]
\end{array}\right\} & \text{Lorentz transf.} \\
d = t\partial_t + s\partial_s + x\partial_x & \text{relat. dilatation} \\
\left.\begin{array}{l}
K_0 = \sqrt{2}\,[t^2\partial_t + x(t-s)\partial_x - s^2\partial_s] - \frac{x^2}{\sqrt{2}}(\partial_s - \partial_t) \\
K_1 = xt\partial_t + (\frac{x^2}{2} - ts)\partial_x + xs\partial_s \\
K_2 = \sqrt{2}\,[x(t+s)\partial_x + s^2\partial_s + t^2\partial_t] - \frac{x^2}{\sqrt{2}}(\partial_s + \partial_t)
\end{array}\right\} & \text{conf. transf.}
\end{array}
\tag{8.2}
$$

The X_i in (3.13) provide just another basis of this algebra :

$$
\begin{array}{ll}
X_1 = -P_x & \text{space translation} \\[6pt]
X_0 = \frac{1}{\sqrt{2}}(P_y - P_0) & \text{time translation} \\
X_2 = -\frac{1}{\sqrt{2}}(P_y + P_0) & \text{vertical translation} \\
X_3 = \frac{1}{\sqrt{2}}\,(M_{01} - M_{12}) & \text{galilean boost} \\
X_4 = \frac{1}{2}\,(M_{02} + d) & \text{non-relat. dilation} \\
X_5 = \frac{1}{2\sqrt{2}}\,(K_0 + K_2) & \text{expansion} \\
X_6 = M_{02} & \text{time dilation} \\
X_7 = \frac{1}{\sqrt{2}}(M_{01} + M_{12}) & \text{"antiboosts"} \\
X_8 = \frac{1}{2\sqrt{2}}(K_0 - K_2) & \mathrm{C}_1 \\
X_9 = K_1 & \mathrm{C}_2
\end{array}
\tag{8.3}
$$

The Bargmann-space transformations constructed above are indeed conformal, $f^* g_{\mu\nu} = \Omega^2 g_{\mu\nu}$. The non–trivial values of the conformal factors are

$$
\Omega \quad = \quad \begin{cases}
e^{\lambda/2} & \text{non-relat. dilatation} \\
\dfrac{1}{1-\kappa t} & \text{expansion} \\
\dfrac{1}{1+\epsilon_1 s} & \mathrm{C}_1 \\
\dfrac{1}{(1-\frac{1}{2}\epsilon_2 x)^2 + \frac{1}{2}\epsilon_2^2 t s} & \mathrm{C}_2
\end{cases}
\tag{8.4}
$$

The factor Ω associated to the two non-relativistic conformal transformations (dilatations and expansions) depends on t only, while those associated to C_1 and C_2 also depend on the other variables.

9 Appendix B : Implementing on fields

The field-dependent action of the two relativistic conformal transformations C_1 and C_2 on space-time is

$$\begin{aligned}
&x^\star = x(1+\epsilon_1\Theta(x^\star,t^\star)),\\
&t^\star = t+\tfrac{1}{2}\epsilon_1 x^2(1+\epsilon_1\Theta(x^\star,t^\star)), \qquad\qquad \mathrm{C}_1\\
&\Theta^\star(x,t) = \frac{\Theta(x^\star,t^\star)}{1+\epsilon_1\Theta(x^\star,t^\star)};
\end{aligned}$$

$$\begin{aligned}
&x^\star = \frac{x+\epsilon_2 t\Theta(x^\star,t^\star)}{1-\frac{1}{2}\epsilon_2 x},\\
&t^\star = \frac{t+\frac{1}{2}\epsilon_2^2 t^2\Theta(x^\star,t^\star)}{(1-\frac{1}{2}\epsilon_2 x)^2}, \qquad\qquad \mathrm{C}_2\\
&\Theta^\star(x,t) = \frac{(1-\frac{1}{2}\epsilon_2 x)^2\Theta(x^\star,t^\star)}{1+\frac{1}{2}\epsilon_2^2\, t\Theta(x^\star,t^\star)}.
\end{aligned} \tag{9.1}$$

The formula for C_1 is consistent with (3.10), since

$$\frac{1}{1-\epsilon_1\Theta^\star(x,t)} = 1+\epsilon_1\Theta(x^\star,t^\star).$$

These transformations are implemented on the density field R according to

$$\begin{aligned}
R^\star(x,t) &= (1+\epsilon_1\Theta(x^\star,t^\star))^4\,\frac{R(x^\star,t^\star)}{J_1^\star}, \qquad \mathrm{C}_1\\
R^\star(x,t) &= \frac{(1+\frac{1}{2}\epsilon_2^2 t\Theta(x^\star,t^\star))^4}{(1-\frac{1}{2}\epsilon_2 x)^8}\,\frac{R(x^\star,t^\star)}{J_2^\star}, \qquad \mathrm{C}_2
\end{aligned} \tag{9.2}$$

where the Jacobians are

$$\begin{aligned}
J_1^\star &= \frac{1+\epsilon_1\Theta}{1-\epsilon_1 x\partial_{x^\star}\Theta-\frac{1}{2}\epsilon_1^2 x^2\partial_{t^\star}\Theta},\\
J_2^\star &= \frac{1+\frac{1}{2}\epsilon_2^2 t\Theta}{\Big((1-\frac{1}{2}\epsilon_2 x)^2-\epsilon_2 t(1-\frac{1}{2}\epsilon_2 x)\partial_{x^\star}\Theta-\frac{1}{2}\epsilon_2^2 t^2\partial_{t^\star}\Theta\Big)\,(1-\frac{1}{2}\epsilon_2 x)^2}.
\end{aligned} \tag{9.3}$$

(In these formulae, Θ means $\Theta(x^\star,t^\star)$).

10 Appendix C : Symmetries of the Schrödinger equation

The potential term (7.6) is manifestly *not* invariant,

$$\frac{1}{8}\frac{\nabla_\mu\rho\nabla^\mu\rho}{\rho} \;\to\; \frac{1}{8}\frac{\nabla_\mu\widetilde{\rho}\nabla^\mu\widetilde{\rho}}{\widetilde{\rho}} + \left(\frac{d^2}{8\Omega^2}\Big[\widetilde{\rho}\nabla_\mu\Omega\nabla^\mu\Omega\Big] - \frac{d}{4\Omega}\Big[\nabla_\mu\Omega\nabla^\mu\widetilde{\rho}\Big]\right).$$

However, the scalar curvature transforms as [16]

$$\mathcal{R} \;\to\; \Omega^{-2}\Big[\mathcal{R} - 2(d+1)\Omega^{-1}\nabla_\mu\nabla^\mu\Omega + (d+1)(2-d)\Omega^{-2}\nabla_\mu\Omega\nabla^\mu\Omega\Big].$$

Therefore, modifying the Lagrangian by adding a term which involves the scalar curvature $\mathcal{R}$ as in Eq. (7.7) allows us to absorbe the symmetry-breaking terms coming from the potential into those which come from transforming $\mathcal{R}$, leaving us with a surface term,

$$\overline{\mathcal{L}}\sqrt{-g} \quad\to\quad \overline{\mathcal{L}}\sqrt{-g} \;-\; \nabla_\mu\Big(\frac{d}{4\Omega}\nabla^\mu\Omega\widetilde{\rho}\sqrt{-g}\Big).$$

The conserved energy-momentum tensor for the linear Schrödinger equation in $(1+1)$ dimensions is found as

$$\begin{gathered}\mathcal{T}_{\mu\nu} = \rho\nabla_\mu\theta\nabla_\nu\theta - \frac{1}{2}g_{\mu\nu}\rho\nabla_\sigma\theta\nabla^\sigma\theta \\ +\frac{\rho}{4}g_{\mu\nu}\nabla_\sigma\theta\nabla^\sigma\theta + \frac{1}{4}\frac{\nabla_\mu\rho\nabla_\nu\rho}{\rho} - \frac{1}{16}g_{\mu\nu}\frac{\nabla_\sigma\rho\nabla^\sigma\rho}{\rho} \\ -\tfrac{1}{8}\nabla_\mu\nabla_\nu\rho + \tfrac{1}{8}\rho\Big(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{4}g_{\mu\nu}\Big).\end{gathered} \tag{10.1}$$

The first line here is the free expression $\mathcal{T}^0_{\mu\nu}$ in (6.15); the second line represents the contribution of the effective potential; the last line comes from the curvature term. Remarkably, this latter term contributes even when *initially* $\mathcal{R} = 0$, since the term $-\frac{1}{8}\nabla_\mu\nabla_\nu\rho$ is present even it such case.